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ABSTRACT

The expressions for kinetic and strain energy for longitudinal stress waves in a bar are considered, first in a one-dimensional model in which cross sections are assumed to remain plane, and stresses in the radial and circumferential direction are assumed to be zero. From this, an equation of motion is derived which is used to determine the speed of longitudinal sinusoidal stress waves as a function of wave length. Secondly, a simplified three-dimensional model is considered where the axial motion is a parabolic function of the radius, from which the speed of sinusoidal stress waves is derived. The derived expressions are compared with previously published solutions.

INTRODUCTION

In the simplest theory of longitudinal stress waves in bars, in which only axial stress and strain is considered, the following equation is obtained:[1]

$$E \frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2} \quad . \tag{1}$$

Rayleigh[2] considers a correction to take into account the kinetic energy of lateral motion, which he states as:

$$T_2 = \pi \rho \int_0^L \int_0^a \dot{v}^2 \, dx \, r \, dr \quad . \tag{2}$$

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He then assumes that

$$v = -\nu \frac{\partial u}{\partial x} r . \tag{3}$$

This is true if $\sigma_r = \sigma_\theta = 0$, for then from Hooke's law,

$$\epsilon_x \equiv \frac{\partial u}{\partial x} = \frac{\sigma_x}{E}$$

and

$$\epsilon_r \equiv \frac{\partial v}{\partial r} = -\nu \frac{\sigma_x}{E} = -\nu \frac{\partial u}{\partial x}$$

which, by assuming that $\partial u/\partial x$ is not a function of r (i.e. plane waves), integrates to:

$$v = -\nu r \frac{\partial u}{\partial x} + c_1 .$$

But when $r = 0$, $v = 0$ and therefore $c_1 = 0$. Thus we obtain expression (3). Differentiating (3) with respect to t we get:

$$\frac{\partial v}{\partial t} = -\nu r \frac{\partial^2 u}{\partial t \, \partial x} . \tag{4}$$

If we substitute (4) into (2),

$$T_2 = \pi \rho \int_0^L \int_0^a \nu^2 r^2 \left(\frac{\partial^2 u}{\partial x \, \partial t} \right)^2 r \, dr \, dx ,$$

We next integrate with respect to r

$$T_2 = \pi \rho \int_0^L \frac{\nu^2 a^4}{4} \left(\frac{\partial^2 u}{\partial t \, \partial x} \right)^2 dx .$$

For a round bar, $\pi a^2 = A$, and $a^2 = 2k^2$, then:

$$T_2 = \frac{\rho A \nu^2 k^2}{2} \int_0^L \left(\frac{\partial^2 u}{\partial t \, \partial x} \right)^2 dx . \tag{5}$$

The correction factor for the frequency of natural vibrations leads to the following expression for the wave speed c of a longitudinal sinusoidal stress wave when $a/\Lambda \ll 1$:

$$c = \sqrt{\frac{E}{\rho}} \left(1 - \frac{\pi^2 \nu^2 a^2}{\Lambda^2} \right) . \qquad (6)$$

Love[3] used the same energy correction for lateral inertia (5) to obtain a differential equation of motion for longitudinal compression waves in bars:

$$E \frac{\partial^2 u}{\partial x^2} = \rho \left(\frac{\partial^2 u}{\partial t^2} - \nu^2 k^2 \frac{\partial^4 u}{\partial x^2 \partial t^2} \right) . \qquad (7)$$

From Love's equation, it may be deduced that for longitudinal sinusoidal waves,[4]

$$c = \sqrt{\frac{E}{\rho} \frac{1}{1 + 2\pi^2 \nu^2 (\frac{a}{\Lambda})^2}} . \qquad (8)$$

This is plotted as curve II of Fig. 1. When $a/\Lambda \ll 1$, (8) reduces to (6). Thus, Love's equation predicts that the shorter the wave length, the lower the speed, which is not in accordance with observed data,[5] nor with the Pochhammer exact solution,[4] which is plotted as curve I of Fig. 1. The exact dependence of wave speed c on wave length is not in closed form, so that numerical results are not easy to obtain, nor is the solution in a form for solving problems of impact, etc.

It will be shown, that with the Rayleigh assumptions, we may add to the expression of strain energy a term representing the shear strain energy of the bar which will give an improved estimate for the velocity of propagation of longitudinal sinusoidal stress waves.

THE ENERGY EXPRESSION FOR THE ONE-DIMENSIONAL THEORY

We utilize the method that Love used to derive (7); that is, obtain complete expressions for T and V that are consistent with the assumptions, and vary (T-V) to obtain the differential equation of axial motion. We make the same assumptions that Rayleigh made: that $\sigma_r = \sigma_\theta = 0$, and plane sections remain plane, or

$$\text{(a)} \quad v = -\nu r \frac{\partial u}{\partial x} , \qquad \text{(b)} \quad u = u(x, t) . \tag{9}$$

Now,

$$\gamma_{xr} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial r} .$$

But $\partial u/\partial r = 0$ from 9(b), and using 9(a) ,

$$\gamma_{xr} = -\nu r \frac{\partial^2 u}{\partial x^2} . \tag{10}$$

The strain energy of shear in a small volume $d\emptyset$ is then

$$d V_2 = \frac{\tau_{xr}}{2} \gamma_{xr} d\emptyset = \frac{G}{2} \gamma^2_{xr} d\emptyset = \frac{G}{2} \nu^2 r^2 \left(\frac{\partial^2 u}{\partial x^2} \right)^2 d\emptyset .$$

If we integrate over the entire bar,

$$V_2 = \int_0^L \int_0^a \frac{G}{2} \nu^2 r^2 \left(\frac{\partial^2 u}{\partial x^2} \right)^2 2\pi r\, dr\, dx = \frac{G\pi \nu^2 a^4}{4} \int_0^L \left(\frac{\partial^2 u}{\partial x^2} \right)^2 dx ,$$

or:

$$V_2 = \frac{G}{2} A k^2 \nu^2 \int_0^L \left(\frac{\partial^2 u}{\partial x^2} \right)^2 dx . \tag{11}$$

The strain energy of longitudinal compression is:

$$V_1 = \frac{1}{2} \int_{\emptyset} (\sigma_x \epsilon_x + \sigma_\theta \epsilon_\theta + \sigma_r \epsilon_r)\, d\emptyset = \frac{EA}{2} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx . \tag{12}$$

The kinetic energy of longitudinal motion is:

$$T_1 = \int_0^L \int_0^a \rho \pi \left(\frac{\partial u}{\partial t} \right)^2 r\, dr\, dx = \int_0^L \rho \frac{\pi a^2}{2} \left(\frac{\partial u}{\partial t} \right)^2 dx$$

or

$$T_1 = \frac{A}{2} \rho \int_0^L \left(\frac{\partial u}{\partial t}\right)^2 dx \quad . \tag{13}$$

The variational equation of motion is

$$\delta \int_0^t dt(T-V) = 0 \, ,$$

or

$$\delta \int_0^t dt(T_1 + T_2 - V_1 - V_2) = 0 \, . \tag{14}$$

By substituting expressions (5), (11), (12), and (13) into (14), we obtain:

$$\delta \int_0^t dt \int_0^L dx \left\{ \frac{A}{2} \rho \left(\frac{\partial u}{\partial t}\right)^2 + \frac{A}{2} \rho \nu^2 k^2 \left(\frac{\partial^2 u}{\partial t \, \partial x}\right)^2 - \frac{EA}{2} \left(\frac{\partial u}{\partial x}\right)^2 \right.$$

$$\left. - \frac{GA}{2} k^2 \nu^2 \left(\frac{\partial^2 u}{\partial x^2}\right)^2 \right\} = 0 \, , \tag{15}$$

and integrating the terms in (15) by parts,

$$\int_0^t \delta \left(\frac{\partial u}{\partial t}\right)^2 dt = 2 \int_0^t \frac{\partial u}{\partial t} \frac{\partial \delta u}{\partial t} dt = 2 \left. \frac{\partial u}{\partial t} \delta u \right]_0^t - 2 \int_0^t \frac{\partial^2 u}{\partial t^2} \delta u \, dt$$

$$= - 2 \int_0^t \frac{\partial^2 u}{\partial t^2} \delta u \, dt \quad . \tag{15a}$$

$$\delta \int_0^t \int_0^L \left(\frac{\partial^2 u}{\partial t \, \partial x}\right)^2 dt \, dx = 2 \int_0^t \int_0^L \frac{\partial^2 u}{\partial t \, \partial x} \frac{\partial^2 \delta u}{\partial t \, \partial x} dt \, dx$$

$$= 2 \int_0^L \left. \frac{\partial^2 u}{\partial t \, \partial x} \frac{\partial \delta u}{\partial x} \right]_0^t dx - 2 \int_0^t \int_0^L \frac{\partial^3 u}{\partial t^2 \, \partial x} \frac{\partial \delta u}{\partial x} dt \, dx$$

$$= -2 \int_0^t \frac{\partial^3 u}{\partial t^2 \partial x} \delta u \Big]_0^L dt + 2 \int_0^t \int_0^L \frac{\partial^4 u}{\partial t^2 \partial x^2} \delta u \, dt \, dx$$

$$= 2 \int_0^t \int_0^L \frac{\partial^4 u}{\partial t^2 \partial x^2} \delta u \, dt \, dx \, . \tag{15b}$$

$$\delta \int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx = 2 \int_0^L \frac{\partial u}{\partial x} \frac{\partial \delta u}{\partial x} dx = 2 \frac{\partial u}{\partial x} \delta u \Big]_0^L - 2 \int_0^L \frac{\partial^2 u}{\partial x^2} \delta u \, dx =$$

$$= 2 \int_0^L \frac{\partial^2 u}{\partial x^2} \delta u \, dx \, . \tag{15c}$$

$$\delta \int_0^L \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx = 2 \int_0^L \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 \delta u}{\partial x^2} dx = 2 \int_0^L \frac{\partial^4 u}{\partial x^4} \delta u \, dx \, . \tag{15d}$$

If we make the indicated substitutions into (15) and equate the coefficient of δu to zero, we obtain the differential equation of axial motion:

$$-\rho \frac{\partial^2 u}{\partial t^2} + \rho \nu^2 k^2 \frac{\partial^4 u}{\partial t^2 \partial x^2} + E \frac{\partial^2 u}{\partial x^2} - G k^2 \nu^2 \frac{\partial^4 u}{\partial x^4} = 0 \, . \tag{16}$$

ANALYSIS OF ONE-DIMENSIONAL SINUSOIDAL STRESS WAVES

To find the relationship between the phase speed c on the wave length Λ, consider a sinusoidal wave:

$$u = D \cos\left(\frac{\omega}{c} x - \omega t\right) \tag{17}$$

and substitute (17) in (16) to obtain:

$$\rho \omega^2 + \rho \nu^2 k^2 \frac{\omega^4}{c^2} - E \frac{\omega^2}{c^2} - G k^2 \nu^2 \frac{\omega^4}{c^4} = 0$$

or:

$$\rho + \rho \nu^2 k^2 \frac{\omega^2}{c^2} - \frac{E}{c^2} - G k^2 \nu^2 \frac{\omega^2}{c^4} = 0 \; . \tag{17a}$$

Now, substituting $\omega = 2\pi c/\Lambda$ into (17a):

$$\rho + \rho \nu^2 k^2 \frac{4\pi^2}{\Lambda^2} - \frac{E}{c^2} - G k^2 \nu^2 \frac{4\pi^2}{\Lambda^2 c^2} = 0 \; . \tag{17b}$$

Since

$$G = \frac{E}{2(1+\nu)} \; ,$$

we obtain

$$\rho \left(1 + \frac{\nu^2 k^2 4\pi^2}{\Lambda^2}\right) = \frac{E}{c^2} \left(1 + \frac{k^2 \nu^2 2\pi^2}{\Lambda^2}\right) \; . \tag{17c}$$

Finally, for a round bar, $k^2 = a^2/2$, therefore

$$\frac{c^2}{E/\rho} = \frac{1 + \frac{\nu^2 \pi^2}{1+\nu}\left(\frac{a}{\Lambda}\right)^2}{1 + 2\nu^2\pi^2\left(\frac{a}{\Lambda}\right)^2} \; . \tag{18}$$

Curve III of Fig. 1 is a plot of expression (18) for $\nu = 0.29$.

With Eq. (18), the asymptotic value of c as $a/\Lambda \to \infty$ is

$$c = \sqrt{\frac{E}{\rho} \frac{\nu^2 \pi^2}{2\nu^2\pi^2(1+\nu)}} = \sqrt{\frac{E}{\rho} \frac{1}{2(1+\nu)}} = \sqrt{\frac{G}{\rho}} = 0.621 \; \frac{E}{\rho} \; , \tag{19}$$

which is the wave speed of distortional waves in isotropic media, but it should be the speed of Rayleigh surface waves,[6] which, for $\nu = 0.29$, is

$$c = 0.9258 \sqrt{\frac{G}{\rho}} = 0.575 \sqrt{\frac{E}{\rho}} \; .$$

The lack of fit of the Love-Rayleigh curve II to the Pochhammer curve I is due to the lack of consideration of shear strain; while the lack of fit of the derived curve III is due to the inaccuracy of the two assumptions by which (18) was obtained. It is obvious that assumption (9a) is inaccurate because the radial and circumferential stresses are not zero for short wave lengths; and (9b) is inaccurate because the axial displacements of waves of short wave length are larger near the surface, but the latter assumption requires plane sections to remain plane.

Mindlin and Herrmann[7] have obtained an expression for c that considers the effect of shear strain for a one-dimensional theory, and this is plotted as curve IV in Fig. 1. They obtained their results by expressing Newton's law in the radial and longitudinal direction in terms of u and v. By assuming (i) v is proportional to r and (ii) plane sections remain plane, and then integrating the expressions over r, two coupled equations are obtained in terms of u and v, which are:

$$\left.\begin{aligned} a^2 K^2 \mu \frac{\partial^2 v_1}{\partial x^2} - 8 K_1^2 (\lambda+\mu) - 4 a K_1^2 \lambda \frac{\partial u}{\partial x} &= \rho a^2 \ddot{v}_1 \\ 2 a \lambda \frac{\partial v_1}{\partial x} + a^2 (\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} &= \rho a^2 \ddot{u} \; ; \end{aligned}\right\} \tag{20}$$

λ and μ are Lamé's constants; K and K_1 are abitrary constants introduced to improve the fit of curve IV; and $v_1 = (a/r)\, v$. Then v_1 is eliminated from the two expressions to obtain the following equation, containing only u :

$$\begin{aligned} &\frac{E}{\rho} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} \\ &\qquad + \frac{a^2 \rho}{8 K_1^2 (\lambda+\mu)} \frac{\partial^2}{\partial t^2} \left[\frac{\lambda + 2\mu}{\rho} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} \right] \\ &\qquad - \frac{a^2 K^2 \mu}{8 K_1^2 (\lambda+\mu)} \frac{\partial^2}{\partial x^2} \left[\frac{\lambda + 2\mu}{\rho} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} \right] = 0 \, . \end{aligned} \tag{21}$$

The discrepancy between the Mindlin and Herrmann curve IV and curve I can be attributed to the assumption that plane sections remain plane, and to an inaccuracy in satisfying the boundary conditions. If expression (17) is substituted into the second of Eqs. (20) we find that:

$$v = \frac{r\pi}{\Lambda} D \left(1 + \frac{2\mu - \rho c^2}{\lambda}\right) \sin \frac{2\pi}{\Lambda} (x - ct) \quad . \tag{22}$$

Since

$$\sigma_r = \lambda \left(\frac{v}{r} + \frac{\partial v}{\partial r} + \frac{\partial u}{\partial x}\right) + 2\mu \frac{\partial v}{\partial r} \quad ,$$

by substitution we obtain

$$\sigma_r = \left(6\mu - 2\rho c^2 + \frac{4\mu - 2\rho c^2}{\lambda/\mu}\right) \frac{\pi}{\Lambda} D \sin \frac{2\pi}{\Lambda} (x - ct) \tag{23}$$

or

$$\sigma_r = \frac{\rho}{\nu} \left(\frac{E}{\rho} - c^2\right) \frac{\pi}{\Lambda} D \sin \frac{2\pi}{\Lambda} (x - ct) \quad . \tag{24}$$

Thus σ_r is not zero on the lateral surfaces of the bar; nor is τ_{rx}.

Mindlin and Herrmann state that if the Rayleigh assumption (9a) is put into (21), it reduces to (7). Equations (20) also include (16) if condition (9a) is substituted. From (9a)

$$v = -\nu r \frac{\partial u}{\partial x} = \frac{r}{a} v_1$$

so that

$$v_1 = -\nu a \frac{\partial u}{\partial x} \quad . \tag{25}$$

If we substitute (25) into (20) and indicate derivatives by dots and primes,

$$\left.\begin{aligned} -a^3 K^2 \mu \nu u''' + 4 K_1^2 a \left[\lambda (2\nu - 1) + 2\nu\mu\right] u' &= \nu \rho a^3 \ddot{u}' \\ a^2 (-2\lambda\nu + \lambda + 2\mu) u'' &= \rho a^2 \ddot{u} \;. \end{aligned}\right\} \tag{26}$$

Now

$$\lambda(2\nu - 1) + 2\nu\mu = \frac{\nu E (2\nu - 1)}{(1 + \nu)(1 - 2\nu)} + \frac{2\nu E}{2(1 + \nu)} = 0 \tag{27}$$

and

$$-2\lambda\nu + \lambda + 2\mu = \lambda(1 - 2\nu) + 2\mu = \frac{\nu E (1 - 2\nu)}{(1 + \nu)(1 - 2\nu)} + \frac{2E}{2(1 + \nu)} = E \;. \tag{28}$$

If we substitute (27) and (28) into (26) we get

$$\left.\begin{aligned} -K^2\mu\nu u''' &= \nu\rho\ddot{u}' \\ Eu'' &= \rho\ddot{u}\,. \end{aligned}\right\} \tag{29}$$

The second of these is Eq. (1). If the first of these equations is multiplied by νk^2, then differentiated with respect to x, and added to the second equation, we obtain

$$-\rho\ddot{u} + \rho\nu^2k^2\ddot{u}'' + Eu'' - K^2\mu k^2\nu^2 u'''' = 0\,. \tag{30}$$

Now, if K is set equal to unity, (30) reduces to Eq. (16).

SECOND APPROXIMATION

The assumption that cross sections remain plane as longitudinal stress waves pass is physically unrealistic, for Eq. (10) gives a finite value of shear strain and stress at the surface of the bar. To eliminate these shear strains, it is necessary that the axial displacement be a function of the radius as well as of x and t. Let

$$u = u_o(x, t)\left(1 - \beta\frac{r^2}{a^2}\right) \tag{31}$$

and assume σ_r to be zero. This determines v since

$$\sigma_r = \lambda(\epsilon_x + \epsilon_r + \epsilon_\theta) + 2\mu\epsilon_r\,. \tag{32}$$

Then

$$0 = \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r}\right) + 2\mu\frac{\partial v}{\partial r}\,. \tag{33}$$

Now substitute for u from (31),

$$\left(\frac{1-\nu}{\nu}\right)\frac{\partial v}{\partial r} + \frac{v}{r} = -\frac{\partial u_o}{\partial x}\left(1 - \frac{\beta r^2}{a^2}\right). \tag{34}$$

Solving (34) for v,

$$v = \frac{A_1}{r\,\frac{\nu}{1-\nu}} - \nu \frac{\partial u_o}{\partial x}\left(r - \frac{\beta}{3-2\nu}\,\frac{r^3}{a^2}\right). \tag{35}$$

Since $v = 0$ when $r = 0$, $A_1 = 0$. To determine β, we use the condition that the shear strain at the surface is zero:

$$\gamma = \frac{\partial u}{\partial r} + \frac{\partial v}{\partial x} = -2\beta\frac{r}{a^2}u_o - \nu^2\frac{\partial^2 u_o}{\partial x^2}\left(r - \frac{\beta r^3}{(3-2\nu)a^2}\right). \tag{36}$$

$$\gamma\Big]_{r=a} = 0 = \frac{-2\beta}{a}u_o - \frac{\nu}{a}\frac{\partial^2 u_o}{\partial x^2} + \frac{\nu\beta a}{(3-2\nu)}\frac{\partial^2 u_o}{\partial x^2}. \tag{37}$$

Hence

$$\beta = \frac{-\nu a^2\frac{\partial^2 u_o}{\partial x^2}}{2u_o - \frac{\nu a^2}{(3-2\nu)}\frac{\partial^2 u_o}{\partial x^2}}. \tag{38}$$

For sinusoidal stress waves, (17), the value of β is:

$$\beta = \frac{4\nu\left(\frac{\pi a}{\Lambda}\right)^2}{2 + \frac{4\nu}{3-2\nu}\left(\frac{\pi a}{\Lambda}\right)^2}. \tag{39}$$

The strain energy of compression is

$$V_1 = \frac{1}{2}\int(\sigma_x\epsilon_x + \sigma_r\epsilon_r + \sigma_\theta\epsilon_\theta)\,d\phi \tag{40}$$

where

$$
\left.\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} = \frac{\partial u_o}{\partial x}\left(1 - \beta\frac{r^2}{a^2}\right) \\
\epsilon_r &= \frac{\partial v}{\partial r} = -\nu\frac{\partial u_o}{\partial x}\left(1 - \frac{3\beta r^2}{(3-2\nu)a^2}\right) \\
\epsilon_\theta &= \frac{v}{r} = -\nu\frac{\partial u_o}{\partial x}\left(1 - \frac{\beta r^2}{(3-2\nu)a^2}\right).
\end{aligned}\right\} \tag{41}
$$

Now

$$
\Delta = \epsilon_x + \epsilon_r + \epsilon_\theta = \frac{\partial u_o}{\partial x}\left[1 + 2\nu + \frac{\beta r^2}{(3-2\nu)a^2}(6\nu - 3)\right]. \tag{42}
$$

For

$$
\sigma_x = \lambda\Delta + 2\mu\epsilon_x\ ; \tag{43}
$$

when $\nu = 0.29$

$$
\sigma_x = E\frac{\partial u_o}{\partial x}\left(1 + 1.0539\frac{\beta r^2}{a^2}\right). \tag{44}
$$

Then

$$
\sigma_x\epsilon_x = E\left(\frac{\partial u_o}{\partial x}\right)^2\left(1 - 1.0539\,\beta\frac{r^2}{a^2}\right)\left(1 - \beta\frac{r^2}{a^2}\right)
$$

$$
= E\left(\frac{\partial u_o}{\partial x}\right)^2\left(1 - 2.0539\,\beta\frac{r^2}{a^2} + 1.0539\,\beta\frac{r^4}{a^4}\right). \tag{45}
$$

Similarly,

$$
\sigma_\theta\epsilon_\theta = E\left(\frac{\partial u_o}{\partial x}\right)^2\left(0.0539\,\beta\frac{r^2}{a^2} - 0.02227\,\beta^2\frac{r^4}{a^4}\right). \tag{46}
$$

Since $\sigma_r = 0$, $\sigma_r\epsilon_r = 0$. Hence

$$
V_1 = \frac{1}{2}\int_o^L\int_o^a E\left(\frac{\partial u_o}{\partial x}\right)^2\left(1 - 2\beta\frac{r^2}{a^2} + 1.0316\,\beta^2\frac{r^4}{a^4}\right)2\pi r\,dr\,dx\ . \tag{47}
$$

We next integrate over r and substitute (31) to get

$$V_1 = \frac{EA}{2}\left(\frac{2\pi}{\Lambda}\right)^2 \psi_1 D^2 \int_o^L \sin^2\left(\frac{2\pi}{\Lambda}\right)(x - ct)\,dx \;; \tag{48}$$

where

$$\psi_1 = 1 - \beta + 0.3439\,\beta^2 . \tag{49}$$

The strain energy of shear is

$$V_2 = \frac{G}{2}\int_o^L\int_o^a \gamma^2\; 2\pi r\,dr\,dx . \tag{50}$$

Since

$$\gamma_{xr} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial r} = \left[-2\beta\frac{r}{a^2} + \nu\left(\frac{2\pi}{\Lambda}\right)^2\left(r - \frac{\beta r^3}{(3-2\nu)a^2}\right)\right] D\cos\frac{2\pi}{\Lambda}(x - ct), \tag{51}$$

$$V_2 = GA\nu^2\left(\frac{2\pi}{\Lambda}\right)^4 \frac{a^2}{4}\,\psi_2 D^2 \int_o^L \cos^2\frac{2\pi}{\Lambda}(x - ct)\,dx , \tag{52}$$

where

$$\psi_2 = 1 - \frac{4}{3}\frac{\beta}{(3-2\nu)} + \frac{\beta^2}{2(3-2\nu)^2} - \frac{\beta}{\nu\pi^2}\left(\frac{\Lambda}{a}\right)^2\left(1 - \frac{2}{3}\frac{\beta}{(3-2\nu)}\right) + \frac{\beta^2}{4\nu^2\pi^4}\left(\frac{\Lambda}{a}\right)^4 . \tag{53}$$

The kinetic energy of longitudinal motion, using (13), is

$$T_1 = \frac{\rho}{2}\left(\frac{2\pi c}{\Lambda}\right)^2 \int_o^L\int_o^a \left(1 - 2\beta\frac{r^2}{a^2} + \beta^2\frac{r^4}{a^4}\right) D^2 \sin^2\frac{2\pi}{\Lambda}(x - ct)\; 2\pi r\,dr\,dx . \tag{54}$$

Integrating with respect to r,

$$T_1 = \frac{\rho}{2}\left(\frac{2\pi c}{\Lambda}\right)^2 A\,\xi_1 D^2 \int_o^L \sin^2\frac{2\pi}{\Lambda}(x - ct)\,dx , \tag{55}$$

where

$$\xi_1 = 1 - \beta + \frac{1}{3}\beta^2 . \tag{56}$$

The kinetic energy of lateral motion, using (2) is

$$T_2 = \frac{\rho}{2} \int_0^L \int_0^a \nu^2 \left(\frac{\partial^2 u_o}{\partial x \partial t}\right)^2 \left(r^2 - \frac{2}{(3-2\nu)} \beta \frac{r^4}{a^2} + \frac{\beta^2}{(3-2\nu)^2} \frac{r^6}{a^4}\right) 2\pi r \, dr \, dx . \tag{57}$$

If we integrate and substitute (31)

$$T_2 = \rho \nu^2 \left(\frac{2\pi}{\Lambda}\right)^4 c^2 A \frac{a^2}{4} \xi_2 D^2 \int_0^L \cos^2 \frac{2\pi}{\Lambda} (x - c\,t)\, dx , \tag{58}$$

where

$$\xi_2 = 1 - \frac{4}{3} \frac{\beta}{(3-2\nu)} + \frac{\beta^2}{2(3-2\nu)^2} . \tag{59}$$

We perform the variation on $T - V$, and obtain

$$\rho\left(\frac{2\pi c}{\Lambda}\right)^2 A \xi_1 D + \rho \nu^2 \left(\frac{2\pi}{\Lambda}\right)^4 c^2 A \frac{a^2}{2} \xi_2 D - E A \left(\frac{2\pi}{\Lambda}\right)^2 \psi_1 D$$

$$- G A \nu^2 \left(\frac{2\pi}{\Lambda}\right)^4 \frac{a^2}{2} \psi_2 D = 0 , \tag{60}$$

from which

$$\frac{c^2}{E/\rho} = \frac{\psi_1 + \frac{\nu^2 \pi^2}{(1+\nu)} \left(\frac{a}{\Lambda}\right)^2 \psi_2}{\xi_1 + 2\nu^2 \pi^2 \left(\frac{a}{\Lambda}\right)^2 \xi_2} . \tag{61}$$

which has the same form as Eq. (18). Expression (61) has been plotted as Curve V in Fig. 1, where it is seen that it fits the exact curve I most closely. The discrepancy between I and V is due to the inaccuracies of assuming a parabolic variation of u with r, and assuming $\sigma_r = 0$. Figure 2 is a comparison of expression (31), the axial displacement, with the exact solution as found by Davies[4] for $a/\Lambda = 0.196$. The radial stress of the exact solution is also plotted.

CONCLUSION

The derived one-dimensional theory of axial motion, expression (16), may be useful in a qualitative way say for determining the dispersion of stress waves; but since this equation is based on the assumption that there is no radial stress and that cross sections remain plane, for greater accuracy the induced curvature of the cross sections should be taken into consideration.

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SUMMARY OF SYMBOLS

a - radius of bar
A - cross-sectional area of bar
c - phase velocity of longitudinal stress waves
D - amplitude of axial displacement
E - Young's modulus
ϵ - elongation
G - shear rigidity, $= E/2(1+\nu)$
k - polar radius of gyration
K, K_1 - arbitrary constants in Mindlin and Herrmann equation
L - length of bar
r - radial coordinate
t - time
T - kinetic energy
T_1 - kinetic energy of longitudinal motion
T_2 - kinetic energy of lateral motion
u - displacement in x (longitudinal) direction
v - displacement in r (radial) direction
V_1 - strain energy of longitudinal strain
V_2 - strain energy of shear
x - longitudinal coordinate

γ - shear strain
Δ - dilatation $= \epsilon_x + \epsilon_r + \epsilon_\theta$
θ - coordinate in circumferential direction
Λ - wave length of sinusoidal waves
λ - Lamé's constant, $= \nu E/(1+\nu)(1-2\nu)$
μ - Lamé's constant, $= G = E/2(1+\nu)$
ν - Poisson's ratio
ξ_1, ξ_2 - correction factors in expression (61)
ρ - mass density
σ - normal stress
τ - shear stress
ϕ - volume
ψ_1, ψ_2 - correction factors in expression (61)
ω - 2π times the frequency

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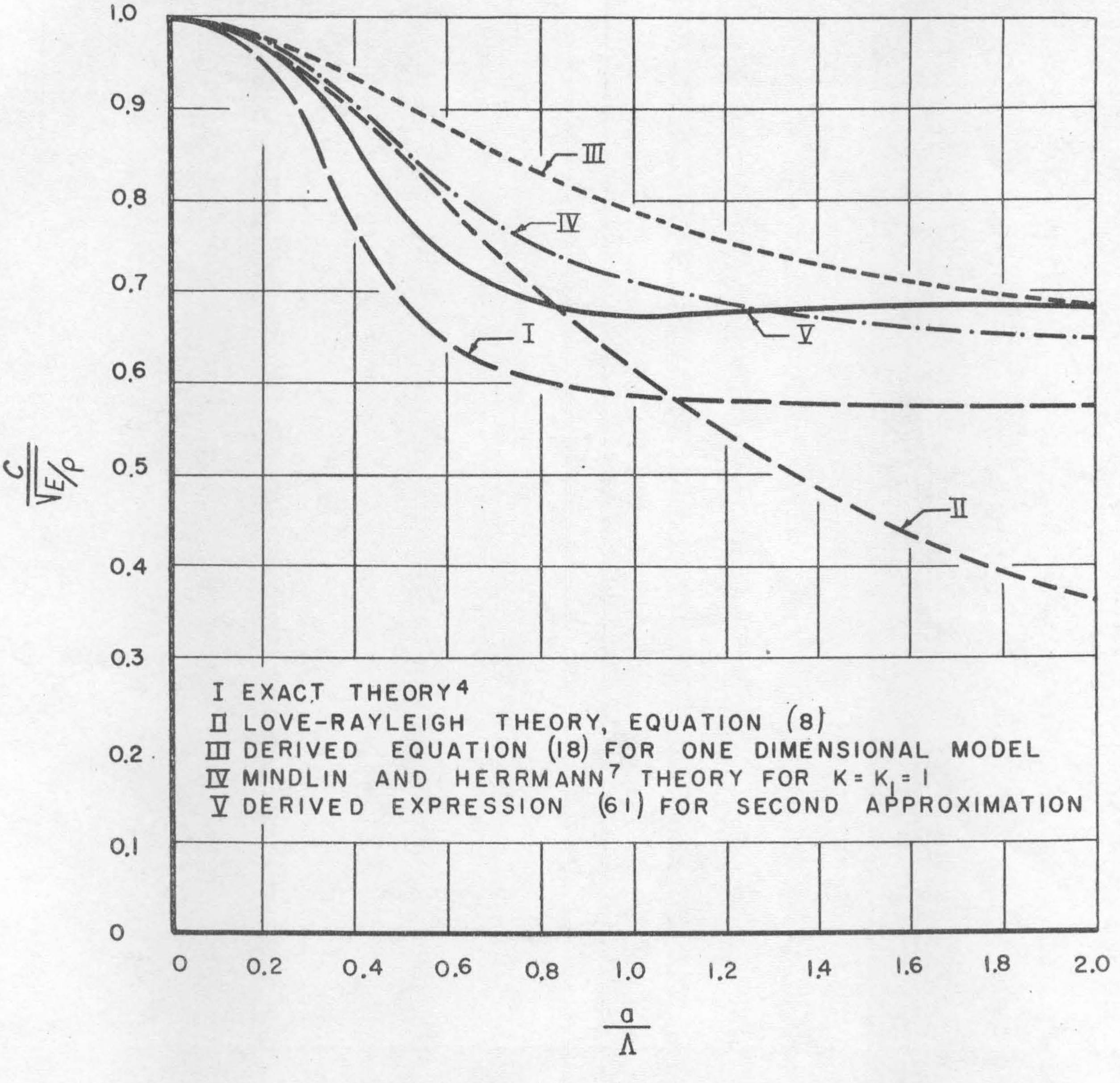

Fig. 1 - Phase velocity of longitudinal stress waves as a function of wave length.

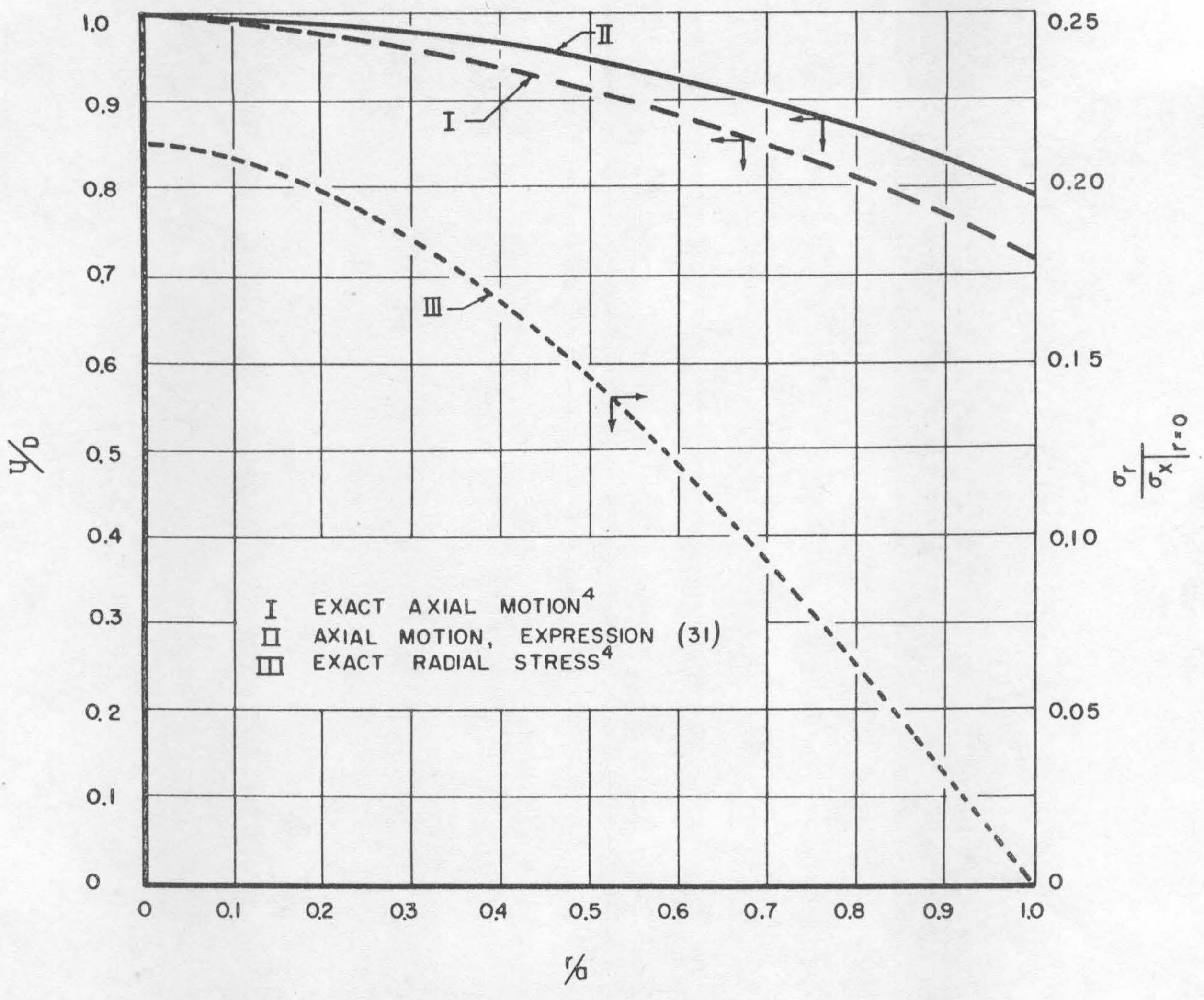

Fig. 2 - Variation of longitudinal motion and radial stress with radius for $a/\Lambda = 0.196$.